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ATTRACTIVITY PROPERTIES OF α-CONTRACTIONS*

by

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It is known that if $T: X \rightarrow X$ is completely continous where X is a Banach space, then point dissipative and compact dissipative are equivalent, and imply the existence of a maximal compact invariant set which is uniformly asymptotically stable and attracts bounded sets uniformly. If T is an contraction it is not known whether point dissipative and compact dissipative are equivalent. However, T is compact dissipative, then there exists a maximal compact invariant set which is uniformly asymptotically stable and →

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SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered) attracts neighborhoods of compact sets uniformly.

Since, in practice, it is much easier to verify that a map is point dissipative rather than compact dissipative, it is desirable to say more about the limiting behavior when T is only assumed to be point dissipative. In this paper, we show, with the addition of only a few general assumptions, that point dissipative and compact dissipative are equivalent. The assumptions seem to be general enough to include almost all of the practical applications. Applications are given, or referenced, to stable neutral functional differential equations, retarded functional differential equations of infinite delay, and strongly damped nonlinear wave equations.

ATTRACTIVITY PROPERTIES OF α-CONTRACTIONS

Paul Massatt

Abstract: It is known that if $T: X \to X$ is completely continuous where X is a Banach space, then point dissipative and compact dissipative are equivalent, and imply the existence of a maximal compact invariant set which is uniformly asymptotically stable and attracts bounded sets uniformly. If T is an α-contraction it is not known whether point dissipative and compact dissipative are equivalent. However, T is compact dissipative, then there exists a maximal compact invariant set which is uniformly asymptotically stable and attracts neighborhoods of compact sets uniformly.

Since, in practice, it is much easier to verify that a map is point dissipative rather than compact dissipative, it is desirable to say more about the limiting behavior when T is only assumed to be point dissipative. In this paper, we show, with the addition of only a few general assumptions, that point dissipative and compact dissipative are equivalent. The assumptions seem to be general enough to include almost all of the practical applications. Applications are given, or referenced, to stable neutral functional differential equations, retarded functional differential equations of infinite delay, and strongly damped nonlinear wave equations.

ATTRACTIVITY PROPERTIES OF α-CONTRACTIONS

Paul Massatt

It is known that if $T: X \to X$ is completely continuous then point dissipative and compact dissipative are equivalent and imply the existence of a maximal compact invariant set which is uniformly asymptotically stable and attracts bounded sets uniformly. If T is an α-contraction it is not known whether point dissipative and compact dissipative are equivalent. However, if T is compact dissipative then there exists a maximum compact invariant set which is uniformly asymptotically stable and attracts neighborhoods of compact sets uniformly (see [2], [4]).

If T is an α-contraction and point dissipative, less is known. Cooperman [2] has shown that under these assumptions, if there is a maximal compact invariant set, then it is uniformly asymptotically stable and attracts neighborhoods of compact sets. This paper will extend some results of a recent paper of mine [8]. In particular, we will show that point dissipative and compact dissipative are equivalent for a large class of α-contractions. In fact, most α-contractions which arise in the applications fall into this class.

Before proceeding, it is best to explain some of the terminology. A bounded set B <u>dissipates</u> a set J under T if there exists n_0 such that $n \geq n_0$ implies $T^n J \subset B$. T is <u>point dissipative</u> if there is a bounded set B which dissipates all points. T is <u>compact dissipative</u> if there is a bounded set B which dissipates all compact sets. T is <u>local dissipative</u> if there is a bounded set which dissipates a neighborhood of any point. T is local compact dissipative if there is a bounded set which dissipates a neighborhood of any compact set. Finally, T is bounded dissipative or ultimately bounded, if there is a bounded set which dissipates bounded sets.

A set J is invariant if $TJ = J$. It is stable if for any $\varepsilon > 0$ there exists a $\delta > 0$ such that for all $n > 0$, $T^n(J+B_\delta(0)) \subset J + B_\varepsilon(0)$ where $B_r(a)$ is a ball of radius r and center a. J attracts B if every neighborhood of J dissipates B. J is uniformly asymptotically stable if it is stable and attracts a neighborhood of itself. The orbit of a set B, $\gamma^+(B)$, is defined by $\gamma^+(B) = \bigcup_{n=0}^{\infty} T^n(B)$ and the ω-limit set of $B, \omega(B)$, is defined by $\omega(B) = \bigcap_{m=0}^{\infty} \mathrm{Cl}\{\bigcup_{n=m}^{\infty} T^n(B)\}$.

The Kuratowski measure of noncompactness, or α-measure, is a useful tool in dealing with a large class of operators which are not compact. The α-measure is a map $\alpha\colon \mathcal{B} \to [0,\infty)$ where $\mathcal{B}$ is the collection of bounded sets, defined by $\alpha(B) = \inf\{r/B$ can be covered by a finite collection of sets of diameter less than $r\}$. In a sense it can be considered a measure of the total boundedness of a set. The α-measure has the following properties:

(i) $\alpha(B) = 0$ if and only if $\mathrm{Cl}\, B$ is compact

(ii) $\alpha(A \cup B) = \max[\alpha(A), \alpha(B)]$

(iii) $\alpha(A + B) \leq \alpha(A) + \alpha(B)$

(iv) $\alpha(\overline{co}\, A) = \alpha(A)$.

T is called an α-contraction if there exists a $k \in [0,1)$ such that for all $B \in \mathcal{B}$ we have $\alpha(TB) \leq k\alpha(B)$. T is a

<u>conditional-α-contraction</u> if there exists a $k \in [0,1)$ such that for all $B \in \mathcal{B}$ with $TB \in \mathcal{B}$ we have $\alpha(TB) \leq k\alpha(B)$. The conditional-$\alpha$-contraction is a more general case which is often more appropriate for the applications.

The major result of this paper is the following theorem.

<u>Theorem</u>: Let X_1 and X_2 be two Banach spaces with $i: X_1 \hookrightarrow X_2$ a compact imbedding. Let $T: X_j \to X_j$ be continuous on both spaces. We assume T can be decomposed as $T = C + U$ with C and U also continuous on X_1 and X_2. Let $C(0) = 0$, C be a contraction on both spaces, and U satisfy the property that for all $B \subset X_1$ with B and $U(B)$ bounded in X_2, then $U(B)$ is bounded in X_1. Under these assumptions point dissipative and compact dissipative are equivalent in X_2, and imply the existence of a maximal compact invariant set in X_2 which is uniformly asymptotically stable, attracts neighborhoods of compact sets, and has a fixed point.

Previously, it was known that point dissipative in X_2 (or X_1) implies bounded dissipative in X_1 [8]. These results above imply much stronger stability results for the space X_2. In particular, we get that point dissipative and compact dissipative are equivalent for all stable neutral functional differential equations and retarded functional differential equations of infinite delay.

The first section of this paper will summarize known results for when T is assumed to be compact dissipative. The second section gives the results mentioned above.

This paper is a part of my Ph.D. thesis [9] at Brown University. I am also deeply grateful to Professor Jack K. Hale for his help

and supervision in the preparation of this paper.

1. Compact Dissipative

In this section, we summarize a few of the known results which apply when T is an α-contraction and compact dissipative. The results apply for more general situations than α-contractions, but for simplicity we confine our attention to α-contractions. These results can be found in [2], [4], [6], [7], and [9].

In the following, let X be a Banach space and $T: X \to X$ be continuous.

Theorem 1: If T is a conditional-α-contraction and $B, \gamma^{+}(B) \subset X$ are bounded, then $\omega(B) \neq \phi$, is compact, invariant , and attracts B.

Theorem 2: If T is a conditional-α-contraction and compact dissipative then there exists a maximal compact invariant set which is uniformly asymptotically stable and attracts neighborhoods of compact sets.

Theorem 3: If T is a conditional-α-contraction and compact dissipative then T has a fixed point.

We now prove that the maximum compact invariant set in the above is connected. We relax the assumptions somewhat to obtain greater generality.

Theorem 4: If K is a compact invariant set which attracts compact sets, then K is connected.

Proof: Clearly $\overline{co}\ K$ is compact and connected. So K attracts $\overline{co}\ K$. Suppose K is not connected. Then there exists open sets U,V with $U \cap K \neq \phi$, $V \cap K \neq \phi$, $K \subset U \cup V$, and $U \cap V = \phi$. By the continuity of T we know $T^n(\overline{co}\ K)$ is connected for each $n \geq 0$. Furthermore, since $K \subset T^n(\overline{co}\ K)$ for all $n \geq 0$ we have $U \cap T^n(\overline{co}\ K) \neq \phi$ and $V \cap T^n(\overline{co}\ K) \neq \phi$. The connectedness of $T^n(\overline{co}\ K)$ implies there is a sequence $\{x_n\}$ with $x_n \in T^n(\overline{co}\ K)$ and $x_n \notin U \cup V$. But K attracts $\{x_n\}$ so $\{x_n\}$ is precompact. Thus $\{x_n\}$ has a converging subsequence with some limit point $x \in K$. Clearly $x \notin U \cup V$ which is a contradiction.

Corollary: The maximal compact invariant set in Theorem 2 is connected.

2. Main Results

Under the assumption that T is a conditional-α-contraction and compact dissipative we can say pretty much about the limiting behavior of $\{T^k\}_{k \geq 0}$. In this section, we give some of the known results when T is a conditional-α-contraction and only known to be/ point dissipative. We will also show for a large class of conditional-α-contractions (which includes most applications) that point dissipative and compact dissipative are equivalent.

The first theorem was originally proved by Cooperman [2]. The second theorem was proved by myself [8]. These two theorems are also contained in my thesis [9].

<u>Theorem 1</u>: If T is a conditional-α-contraction and K is a compact invariant set which attracts points, then the following are equivalent.

(i) K attracts compact sets

(ii) K is stable

(iii) K is a maximal compact invariant set.

One of the important implications of this theorem is given by the following corollary.

<u>Corollary 1</u>: If T is a conditional-α-contraction, point dissipative, and has a maximal compact invariant set A, then A is uniformly asymptotically stable, connected, and attracts neighborhoods of compact sets.

<u>Proof</u>: Theorem 1 implies A attracts compact sets. Hence, T is compact dissipative. Now apply Theorem 2 and Theorem 4 of section 1.

<u>Theorem 2</u>: Let $i: X_1 \hookrightarrow X_2$ be a compact imbedding where X_j are Banach spaces with norm $||\cdot||_j$. Let T, C, and U be continuous operators mapping X_j into itself. Let $T = C + U$ with $C(0) = 0$, C a contraction in X_1 and U having the property that if $B \subset X_1$ and $U(B)$ are bounded in X_2 then $U(B)$ is bounded in X_1. Then the following are equivalent:

(i) T is point dissipative in X_1

(ii) T is bounded dissipative in X_1

(iii) there is a bounded set in X_2 which dissipates points in X_1.

In the next theorem we combine these results to prove the equivalence of point dissipative and compact dissipative under certain natural hypotheses.

Theorem 3: Under the conditions of Theorem 2, and (*) any compact invariant set in X_2 is a subset of the closure in X_2 of a bounded set in X_1, then point dissipative and compact dissipative are equivalent in X_2.

Proof: From Theorem 2, there is a bounded set B in X_1 which dissipates bounded sets in X_1. Then $\gamma^+(B)$ is bounded in X_1, and hence, precompact in X_2. Therefore, its ω-limit set in X_2, $\omega_2(B)$, is nonempty, compact, invariant and attracts B in X_2. Now let J be any compact invariant set in X_2. Then there is a bounded set A in X_1 with $J \subset Cl_2(A)$. Now, since B dissipates bounded sets in X_1, B dissipates A. Hence $\omega_2(B)$ attracts A in X_2. Since T is continuous, $\omega_2(B)$ attracts $Cl_2(A)$ in X_2. Hence, $J \subset Cl_2 A$ implies $\omega_2(B)$ attracts J in X_2. But since J is invariant, $J \subset \omega_2(B)$. Hence, $\omega_2(B)$ is a maximal compact invariant set in X_2. Now apply Corollary 1.

Corollary 2: Under the conditions of Theorem 2, C a contraction in both spaces, and if we know any compact invariant set in X_2 is a subset of the closure in X_2 of a bounded set in X_1, then point dissipative implies there exists a maximal compact invariant set which is connected, uniformly asymptotically stable, attracts a neighborhood of any compact set, and has a fixed point.

Proof: Clearly, T is a conditional-α-contraction in X_2 since C is a contraction and U is conditionally-completely continuous. Also, Theorem 3 implies T is compact dissipative. Hence, Theorems 2 and 3 imply the result.

The condition (*) may not always be easily verified. A more natural and simple condition to verify is given in the next theorem.

Theorem 4: Assume the conditions of Theorem 2 and let C be a contraction in both spaces. Then point dissipative and compact dissipative are equivalent.

Proof: All we need to do is show that any compact invariant set in X_2 is a subset of the closure in X_2 of a bounded set in X_1 and apply Theorem 3. Let J be a compact invariant set in X_2. Let k be a contraction constant for C in both spaces. Let $r = 1/(1-k)$. Let B be a closed ball in X_1 with $rU(J) \subset B$. We will show $J \subset Cl_2(B)$.

Let $d_2(x,B) = \inf_{y \in B} ||x-y||_2$. If $d_2(x,B) = 0$ then $x \in Cl_2(B)$. Let $\eta = \sup\{d_2(x,B)/x \in J\}$. If we show $\eta = 0$ then we have $J \subset Cl_2(B)$. This is our goal.

Let $x \in J$ and $y \in B$. Let $z = Cy + Ux$. Then $||z||_1 \leq k||y||_1 + ||Ux||_1 \leq k||B||_1 + (1/r)||B||_1 \leq ||B||_1$. Hence, $z \in B$. Furthermore, $||Tx-z||_2 = ||(Cx+Ux) - (Cy+Ux)||_2 = ||Cx-Cy||_2 \leq k||x-y||_2$. Hence, $d_2(Tx,B) = \sup_{y \in B} ||Tx-y||_2 \leq k \sup_{y \in B} ||x-y||_2 = k\, d_2(x,B)$. But this implies,

$\eta = \sup_{x\in J} d_2(x,B) = \sup_{x\in J} d_2(Tx,B) \le k \sup_{x\in J} d_2(x,B) = \eta$. In the second step we use the invariance of J. Thus, we get $\eta = 0$ and so the result is proved.

Corollary 3: Under the conditions of Theorem 2 and if C is a contraction in both spaces then point dissipative implies the existence of a maximal compact invariant set which is connected, uniformly asymptotically stable, attracts a neighborhood of any compact set, and has a fixed point.

Proof: The proof of Theorem 4 shows (*) is satisfied. Hence, we may apply Corollary 3.

Remark: These theorems show that for stable neutral functional differential equations, and for retarded functional differential equations of infinite delay with phase space in a Banach space, point dissipative and compact dissipative are equivalent. To verify the hypotheses, see [8]. For an application to strongly damped nonlinear wave equations, see [10].

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